Outline

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Student Learning Objectives SLOs: Eigen-values and vectors: Diagonalization

SLOs 7.1 Eigen-values and vectors: Diagonalization

After this lecture you should know how
- Matrix Diagonalization
- Similarity Transformation
- Eigenvalues, Eigenvectors, and Eigenbases
are inter-related.

\[
S^{-1} \quad A \quad S \quad D
\]

Example: Consider linear transformations \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). What do we know? — They rotate, reflect, and stretch our input space (Left panel) in various ways; two examples shown in the Center and Right panels.

An eigenvector, \( \vec{v} \), of a linear transformation is a vector whose orientation is preserved by the transformation, \( i.e. \vec{v} \parallel A\vec{v} \), \( i.e. \)

\[
A\vec{v} = \lambda \vec{v}
\]

and the scalar \( \lambda \) is the eigenvalue associated with \( \vec{v} \).
Why Should We Care About Eigenvalues?

From a "pure" linear algebra perspective, operations on eigenvectors are easy, since they are just (multiplicative) scalings.

In applications the eigenvector-eigenvalue pair describe some fundamental property of a "system" (something we are using a mathematical model to describe):

- **Vibrations**, either in strings (guitars, pianos, etc) or other structures (bridges, tall buildings): the eigenvalue is the frequency, and the eigenvector is the deformation. [Tacoma Bridge]

  (Link sponsored by “C+ Engineering LLC.”)

- In statistics, **Principal Component Analysis**, is an eigenvector-eigenvalue analysis of the correlation matrix, and is used to study large data sets, such as those encountered in data mining, chemical research, psychology, and in marketing.

**Buzzwords**: “Data Science,” “Big Data,” and “Page Rank.”
Motivating Example

Given the matrices

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix}$$

ponder the “fun” of

- computing:
  - $A^5$, $\text{rank}(A)$, $\text{det}(A)$, the basis of $\ker(A)$, and the basis of $\text{im}(A)$

- computing:
  - $B^5$, $\text{rank}(B)$, $\text{det}(B)$, the basis of $\ker(B)$, and the basis of $\text{im}(B)$

Motivating Example (A)

For Matrix $A$ we can write down the answers quickly:

$$\text{rank}(A) = 3, \quad \text{det}(A) = 0$$

$$A^5 = \begin{bmatrix} (-1)^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1^5 & 0 \\ 0 & 0 & 0 & 2^5 \end{bmatrix} = \begin{bmatrix} (-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$$\ker(A) \in \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \text{im}(A) \in \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Diagonalizable Matrices

Definition (Diagonalizable Matrices)

Consider a linear transformation $T(\vec{x}) = A\vec{x}$; $(T : \mathbb{R}^n \rightarrow \mathbb{R}^n)$. Then $A$ (and/or $T$) is said to be diagonalizable if the matrix $B$ of $T$ with respect to some basis, $\mathcal{B}(\mathbb{R}^n)$ is diagonal.

By previous discussion [Notes#3.4], the matrix $A$ is diagonalizable if and only if it is similar to some diagonal matrix $B$; meaning that there exists some invertible matrix $S$, so that

$$S^{-1}AS = B$$

is a diagonal matrix.

Definition (Diagonalization of a Matrix)

To diagonalize a square matrix $A$ means to find an invertible matrix $S$ and a diagonal matrix $B$ such that $S^{-1}AS = B$. 
Eigenvectors, Eigenvalues, and Eigenbases

Definition (Eigenvectors, Eigenvalues, and Eigenbases)
Consider a linear transformation \( T(x) = Ax \); \( (T : \mathbb{C}^n \mapsto \mathbb{C}^n) \). A non-zero vector \( \vec{v} \in \mathbb{C}^n \) is called an eigenvector of \( A \) (and/or \( T \)) if

\[
A\vec{v} = \lambda \vec{v},
\]

for some \( \lambda \in \mathbb{C} \). This \( \lambda \) is called the eigenvalue associated with the eigenvector \( \vec{v} \).

A basis \( \vec{v}_1, \ldots, \vec{v}_n \) of \( \mathbb{C}^n \) is called an eigenbasis for \( A \) (and/or \( T \)) if the vectors \( \vec{v}_1, \ldots, \vec{v}_n \) are eigenvectors of \( A \), i.e.

\[
A\vec{v}_k = \lambda_k \vec{v}_k, \quad k = 1, \ldots, n
\]

for some scalars \( \lambda_1, \ldots, \lambda_n \).

Example \( A^k \vec{v} \) in \( \mathbb{R}^2 \)

Example (Life in \( \mathbb{R}^2 \))
Assume we have an eigenbasis \( \{\vec{v}_1, \vec{v}_2\} \), then any \( \vec{w} \in \mathbb{R}^2 \) can be written as \( \vec{w} = a_1 \vec{v}_1 + a_2 \vec{v}_2 \) for unique scalars \( a_1 \) and \( a_2 \); now

\[
A\vec{w} = A(a_1 \vec{v}_1 + a_2 \vec{v}_2) = a_1 A\vec{v}_1 + a_2 A\vec{v}_2 = a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_2 \vec{v}_2
\]

doing

\[
A^k \vec{w} = A^{k-1}(A(a_1 \vec{v}_1 + a_2 \vec{v}_2)) = A^{k-1}(a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_2 \vec{v}_2) = a_1 \lambda_1^k \vec{v}_1 + a_2 \lambda_2^k \vec{v}_2
\]

"Future-Proofing:"
\( \mathbb{R}^n \): if \( \vec{v}_1, \vec{v}_2 \) are eigenvectors of \( A \), and \( \vec{w} = a_1 \vec{v}_1 + a_2 \vec{v}_2 \) we have \( A^k \vec{w} = a_1 \lambda_1^k \vec{v}_1 + a_2 \lambda_2^k \vec{v}_2 \). That is the “action” is the “same” as above when restricted to \( V = \text{span} (\vec{v}_1, \vec{v}_2) \).
Example $A^k \vec{v}$ in $\mathbb{R}^2$

Eigenvalues and Eigenvectors
Diagonalization
Suggested Problems

Motivating Example
Definitions, etc...

Eigenbases and Diagonalization

Theorem (Eigenbases and Diagonalization)

The matrix $A$ is diagonalizable if and only if there exists an eigenbasis for $A$. If $\vec{v}_1, \ldots, \vec{v}_n$ is an eigenbasis for $A$, with $A\vec{v}_1 = \lambda_1 \vec{v}_1, \ldots, A\vec{v}_n = \lambda_n \vec{v}_n$, then the matrices

$$S = [\vec{v}_1 \ldots \vec{v}_n], \quad \text{and} \quad B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

will diagonalize $A$, meaning $S^{-1}AS = B$.

Conversely, if the matrices $S$ and $B$ diagonalize $A$, then the columns of $S$ will form an eigenbasis for $A$, and the diagonal entries of $B$ will be the associated eigenvalues.

Example of the Identity Transformation

Example $(T(\vec{x}) = I_n \vec{x} = \vec{x})$

Find all the eigenvalues and eigenvectors of the identity matrix $I_n$.

Solution: Since $I_n \vec{x} = 1\vec{x} \quad \forall \vec{x} \in \mathbb{R}^n,$

it follows that all $\vec{x} \in \mathbb{R}^n$ are eigenvectors of $I_n$ with associated eigenvalues 1.

Therefore, all bases of $\mathbb{R}^n$ are eigenbases for $I_n$, and clearly the already diagonal matrix $I_n$ is diagonalizable. Any invertible $S$ will do the trick: $S^{-1}I_nS = S^{-1}S = I_n$. 

Peter Blomgren (blomgren@sdsu.edu) 7.1. Eigen-values and vectors: Diagonalization — (17/41)

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Peter Blomgren (blomgren@sdsu.edu) 7.1. Eigen-values and vectors: Diagonalization — (20/41)
Eigenvalues of a Projection

Example (Projection: Setup)

Let \( \vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \), and consider the projection onto the line
\( L = \text{span}\{\vec{w}\} \):

\[
T(\vec{x}) = \text{proj}_L(\vec{x}) = \left( \frac{\vec{x} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} = P\vec{x}
\]

where [Notes#2.2] the projection matrix \( P \) is

\[
P = \frac{1}{25} \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}
\]

What About Rotations?

Example (Rotation by \( \pi/2 \) (90°))

Let \( R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \), and \( T(\vec{x}) = R\vec{x} \) be the rotation transformation.

Now, given any \( \vec{x} \in \mathbb{R}^2 \), \( \vec{x} \) is not parallel to \( R\vec{x} \).

As long as we insist on REAL eigenvectors and REAL eigenvalues, we find none...

Matrices with real entries may have Complex eigenvalues.

Complex: The (complex) eigenvalues of the matrix above are \( 0+1i \), and \( 0-1i \); where \( i = \sqrt{-1} \).
Matrices with Eigenvalue 0

Core Property — Zero Eigenvalue

By definition 0 is an eigenvalue if and only if we can find a non-zero \( \vec{x} \in \mathbb{R}^n \) so that \( A\vec{x} = 0\vec{x} = \vec{0} \).

That means 0 is an eigenvalue of \( A \) if and only if \( \ker(A) \neq \{\vec{0}\} \), i.e. \( A \) is non-invertible.

We add this to our list from [Notes#2.4], [Notes#3.1], and [Notes#3.3].
Eigenvalues and Eigenvectors
Diagonalization
Suggested Problems
Suggested Problems 7.1
Lecture – Book Roadmap

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Supplemental Material
Metacognitive Reflection
Problem Statements 7.1
Complex Analysis: Essentials for Linear Algebra

(7.1.1), (7.1.3), (7.1.5), (7.1.7)

(7.1.1) Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix, and $\vec{v}$ an eigenvector of $A$, with associated eigenvalue $\lambda$. Is $\vec{v}$ an eigenvector of $A^3$? If so, what is the eigenvalue?

(7.1.3) Let $A \in \mathbb{R}^{n \times n}$ be an invertible matrix, and $\vec{v}$ an eigenvector of $A$, with associated eigenvalue $\lambda$. Is $\vec{v}$ an eigenvector of $A + 2I_n$? If so, what is the eigenvalue?

(7.1.5) If a vector $\vec{v}$ is an eigenvector of both $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$, is $\vec{v}$ necessarily an eigenvector of $A + B$?

(7.1.7) If $\vec{v}$ is an eigenvector of $A \in \mathbb{R}^{n \times n}$, with eigenvalue $\lambda$, what can you say about $\ker(A - \lambda I_n)$?

(7.1.15), (7.1.17), (7.1.21)

(7.1.15) Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformation — Reflection about a line $L$ in $\mathbb{R}^2$ — find the eigenbasis (if possible) and determine whether the transformation is diagonalizable?

(7.1.17) Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformation — Counterclockwise rotation through an angle of $45^\circ$ ($\pi/4$) followed by a scaling by 2 in $\mathbb{R}^2$ — find the eigenbasis (if possible) and determine whether the transformation is diagonalizable?

(7.1.21) Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformation — Scaling by 5 in $\mathbb{R}^3$ — find the eigenbasis (if possible) and determine whether the transformation is diagonalizable?

Supplemental Material
Metacognitive Exercise — Thinking About Thinking & Learning
I know / learned | Almost there | Huh?!? 
Right After Lecture

After Thinking / Office Hours / SI-session

After Reviewing for Quiz/Midterm/Final
Complex Multiplication

Definition (Complex Multiplication)
Let $z_1, z_2 \in \mathbb{C}$, then
\[
z_1z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)
\]
this follows from the fact that $i^2 = -1$.

Note: $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$
Let $T : \mathbb{R}^2 \mapsto \mathbb{C}$ be the linear transformation:
\[
T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = a + ib, \quad T^{-1}(a + ib) = \begin{bmatrix} a \\ b \end{bmatrix},
\]
that is we can interpret vectors in $\mathbb{R}^2$ as complex numbers (and the other way around).

Complex Conjugate

Definition (Complex Conjugate)
Given $z = (a + ib) \in \mathbb{C}$, the complex conjugate is defined by
\[
z = (a - ib), \quad \text{sometimes } z^* = (a - ib)
\]
(reversing the sign on the imaginary part). Note that this is a reflection across the real axis in the complex plane.

Hey! It’s a reflection across the real axis!
$z$ and $z^*$ form a **conjugate pair** of complex numbers, and
\[
z z^* = (a + ib)(a - ib) = a^2 + b^2.
\]
Polar Coordinate Representation

Polar Coordinate Representation (Modulus and Argument)
We can represent \( z = a + ib \) in terms of its length \( r \) (modulus) and angle \( \theta \) (argument); where

\[ r = \text{mod}(z) = |z| = \sqrt{a^2 + b^2}, \quad \theta = \arg(z) \in [0, 2\pi) \]

where

\[ \theta = \arg(z) = \begin{cases} \arctan(\frac{b}{a}) & \text{if } a > 0 \\ \arctan(\frac{b}{a}) + \pi & \text{if } a < 0 \text{ and } b \geq 0 \\ \arctan(\frac{b}{a}) - \pi & \text{if } a < 0 \text{ and } b < 0 \\ \frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0 \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0 \\ \text{indeterminate} & \text{if } a = 0 \text{ and } b = 0. \end{cases} \]

De Moivre's Formula

\[ z = r(\cos \theta + i \sin \theta) \equiv re^{i\theta}, \]

where the identity

\[ e^{i\theta} = (\cos \theta + i \sin \theta) \]

is known as Euler's Formula.

Once we restrict the range of \( \theta \) to an interval of length \( 2\pi \), the representation is unique. Common choices are \( \theta \in [0, 2\pi) \) [we will use this here], or \( \theta \in [-\pi, \pi) \); but \( \theta \in [\xi, \xi + 2\pi) \) for any \( \xi \in \mathbb{R} \) works (but why make life harder than necessary?!)

Multiplying in Polar Form

**Example**

Given \( z_1, z_2 \in \mathbb{C} \), then

\[ z_1z_2 = \begin{cases} (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \\ r_1e^{i\theta_1}r_2e^{i\theta_2} = (r_1r_2)e^{i(\theta_1 + \theta_2)} \\ r_1(\cos \theta_1 + i \sin \theta_1)r_2(\cos \theta_2 + i \sin \theta_2) = (r_1r_2)((\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)) \end{cases} \]

these three expressions are equivalent.

Since Euler's formula says \( e^{i(\theta_1 + \theta_2)} = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \), we can restate some old painful memories:

\[
\begin{align*}
\cos(\theta_1 + \theta_2) &= \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \\
\sin(\theta_1 + \theta_2) &= \cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2
\end{align*}
\]

Bottom line, for \( z = z_1z_2 \), we have

\[ |z| = |z_1||z_2|, \quad \arg(z) = \arg(z_1) + \arg(z_2) \mod 2\pi. \]

From Euler to De Moivre

From Euler’s Identity \( e^{i\theta} = (\cos \theta + i \sin \theta) \) we see that

\[ (\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i \sin(n\theta), \]

which is known as De Moivre’s Formula.

OK, we have enough fragments of Complex Analysis to state the key result we need prior to revisiting our Eigenvalue/Eigenvector problem space.
Fundamental Theorem of Algebra

Theorem (Fundamental Theorem of Algebra)

Any $n$th degree polynomial $p_n(\lambda)$ with complex coefficients\(^*\) can be written as a product of linear factors

$$p_n(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

for some complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $k$. (The $\lambda_k$'s need not be distinct).

Therefore a polynomial $p_n(\lambda)$ of degree $n$ has precisely $n$ complex roots if they are counted with their multiplicity.

\(^*\) Note that real coefficients are complex coefficients with zero imaginary part.