### Student Learning Objectives

SLOs 7.1 Eigen-values and vectors: Diagonalization

After this lecture you should know how

- Matrix Diagonalization
- Similarity Transformation
- Eigenvalues, Eigenvectors, and Eigenbases are inter-related.

### Introduction

**Figure:** Consider linear transformations \( T : \mathbb{R}^2 \to \mathbb{R}^2 \). What do we know? — They rotate, reflect, and stretch our input space (LEFT PANEL) in various ways; two examples shown in the CENTER and RIGHT PANELS. An **eigenvector**, \( \vec{v} \), of a linear transformation is a vector whose orientation is preserved by the transformation, \( i.e. \vec{v} \parallel A\vec{v} \), \( i.e. \)

\[
A\vec{v} = \lambda \vec{v},
\]

and the scalar \( \lambda \) is the **eigenvalue** associated with \( \vec{v} \).
Why Should We Care About Eigenvalues?

From a “pure” linear algebra perspective, operations on eigenvectors are easy, since they are just (multiplicative) scalings. In applications the eigenvector-eigenvalue pair describe some fundamental property of a “system” (something we are using a mathematical model to describe):

- **Vibrations**, either in strings (guitars, pianos, etc) or other structures (bridges, tall buildings): the eigenvalue is the frequency, and the eigenvector is the deformation. [Tacoma Bridge]

- In statistics, **Principal Component Analysis**, is an eigenvector-eigenvalue analysis of the correlation matrix, and is used to study large data sets, such as those encountered in data mining, chemical research, psychology, and in marketing. **Buzzwords**: “Big Data,” and “Page Rank.”

**Applications**

- **Cooking**

  Figure: “Eigenvalue Analysis of Microwave Oven.” Haider, Siddique, Abbas, and Ahmed — **INTERNATIONAL JOURNAL OF SCIENTIFIC & ENGINEERING RESEARCH, VOLUME 4, ISSUE 9, SEPTEMBER-2013, PP. 2473.**

- **Combustion**

  Figure: Space and time evolution of a three-cells hopping state found in simulations of the Kuramoto-Sivashinsky equation. The cells move non-uniformly and their shapes change periodically. Parameter values are: \( \varepsilon = 0.32, \eta_1 = 1.0, \eta_2 = 0.013, \) and \( R = 7.7475. \) — Blomgren, Gasner, and Palacios, “Hopping Behavior in the Kuramoto-Sivashinsky Equation.” **CHAOS: AN INTERDISCIPLINARY JOURNAL OF NONLINEAR SCIENCE, VOLUME 15; MARCH 28, 2005.**

Why Should We Care About Eigenvalues? (continued)

OK, we may not be quite ready to identify the energy states in the hydrogen atom... yet. Let us, momentarily, retreat to our safe Linear Algebra “universe.”

Example: Energy Modes in the Hydrogen Wave Function
Motivating Example

Given the matrices

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 8 & 7 & 6 \\
5 & 4 & 3 & 2
\end{bmatrix}
\]

ponder the “fun” of computing:
- \(A^5, \text{rank}(A), \det(A), \) the basis of \(\ker(A)\), and the basis of \(\text{im}(A)\)
- \(B^5, \text{rank}(B), \det(B), \) the basis of \(\ker(B)\), and the basis of \(\text{im}(B)\)

Motivating Example (A)

For Matrix \(A\) we can write down the answers quickly:

\[
\text{rank}(A) = 3, \quad \det(A) = 0
\]

\[
A^5 = \begin{bmatrix}
(-1)^5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1^5 & 0 \\
0 & 0 & 0 & 2^5
\end{bmatrix} = \begin{bmatrix}
(-1) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 32
\end{bmatrix}
\]

\[
\ker(A) \in \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \text{im}(A) \in \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Diagonalizable Matrices

Definition (Diagonalizable Matrices)

Consider a linear transformation \(T(\vec{x}) = A\vec{x}; (T : \mathbb{R}^n \rightarrow \mathbb{R}^n)\).
Then \(A\) (and/or \(T\)) is said to be diagonalizable if the matrix \(B\) of \(T\) with respect to some basis is diagonal.

By previous discussion (NOTES 3.4), the matrix \(A\) is diagonalizable if and only if it is similar to some diagonal matrix \(B\); meaning that there exists some invertible matrix \(S\), so that

\[
S^{-1}AS = B \quad \text{is a diagonal matrix.}
\]

Definition (Diagonalization of a Matrix)

To diagonalize a square matrix \(A\) means to find an invertible matrix \(S\) and a diagonal matrix \(B\) such that \(S^{-1}AS = B\).
Eigenvectors, Eigenvalues, and Eigenbases

Definition (Eigenvalues, Eigenvalues, and Eigenbases)
Consider a linear transformation $T(x) = Ax$; ($T : \mathbb{R}^n \rightarrow \mathbb{R}^n$). A non-zero vector $\vec{v} \in \mathbb{R}^n$ is called an eigenvector of $A$ (and/or $T$) if

$$A\vec{v} = \lambda \vec{v},$$

for some $\lambda \in \mathbb{R}$. This $\lambda$ is called the eigenvalue associated with the eigenvector $\vec{v}$.

A basis $\vec{v}_1, \ldots, \vec{v}_n$ of $\mathbb{R}^n$ is called an eigenbasis for $A$ (and/or $T$) if the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are eigenvectors of $A$, i.e.

$$A\vec{v}_k = \lambda_k \vec{v}_k, \quad k = 1, \ldots, n$$

for some scalars $\lambda_1, \ldots, \lambda_n$.

Example (Repeated Multiplication by $A$)
If $\vec{v}$ is an eigenvector of $A$, then $A\vec{v} = \lambda \vec{v}$; and

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda \vec{v}) = \lambda A\vec{v} = \lambda^2 \vec{v}.$$ 

If does not take a lot of imagination to realize

$$A^k \vec{v} = \lambda^k \vec{v}$$

Example $A^k \vec{v}$ in $\mathbb{R}^2$

Example (Life in $\mathbb{R}^2$)
Assume we have an eigenbasis $\{\vec{v}_1, \vec{v}_2\}$, then any $\vec{w} \in \mathbb{R}^2$ can be written as $\vec{w} = a_1 \vec{v}_1 + a_2 \vec{v}_2$ for unique scalars $a_1$ and $a_2$; now

$$A\vec{w} = A(a_1 \vec{v}_1 + a_2 \vec{v}_2) = a_1 A\vec{v}_1 + a_2 A\vec{v}_2 = a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_2 \vec{v}_2$$

and

$$A^k \vec{w} = A^{k-1}(A(a_1 \vec{v}_1 + a_2 \vec{v}_2)) = A^{k-1}(a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_2 \vec{v}_2) = a_1 \lambda_1^k \vec{v}_1 + a_2 \lambda_2^k \vec{v}_2$$
Eigenvalues of a Projection

Example (Projection: Setup)

Let \( \vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \), and consider the projection onto the line \( L = \text{span} \{ \vec{w} \} \):

\[
T(\vec{x}) = \text{proj}_L(\vec{x}) = \frac{\vec{x} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} = P\vec{x}
\]

where (NOTES 2.2)

\[
P = \frac{1}{25} \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}
\]
Eigenvalues of a Projection

Example (Projection: Getting Started...)

(Restating old results in this new context —) Any vector \( \parallel \) to \( L (\vec{w}) \) will be projected onto itself (hence it is an eigenvector with eigenvalue 1); and any vector \( \perp \) to \( L (\vec{w}) \) will be projected onto \( 0 \), so it is an eigenvector with eigenvalue 0;

\[
P\vec{x}^\parallel = \vec{x}^\parallel, \quad P\vec{x}^\perp = 0\vec{x}^\perp
\]

one eigenbasis is

\[
\mathcal{B} = (\vec{x}^\parallel, \vec{x}^\perp), \quad \text{where} \quad \vec{x}^\parallel = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \vec{x}^\perp = \begin{bmatrix} -3 \\ 4 \end{bmatrix},
\]

What About Rotations?

Example (Rotation by \( \pi/2 \) (90°))

Let \( R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \), and \( T(\vec{x}) = R\vec{x} \) be the rotation transformation.

Now, given any \( \vec{x} \in \mathbb{R}^2 \), \( \vec{x} \) is not parallel to \( R\vec{x} \).

As long as we insist in REAL eigenvectors and REAL eigenvalues, we find none... Complex eigenvalues will be considered soon.

Flash-Forward: The eigenvalues of the matrix above are \( 0 + 1i \) and \( 0 - 1i \); where \( i = \sqrt{-1} \).

Eigenvalues of Orthogonal Matrices

Example (Orthogonal \( A \in \mathbb{R}^{n \times n} \))

Let \( A \) be an orthogonal matrix; then \( T(\vec{x}) = A\vec{x} \) preserves length, so if/when \( \vec{v} \) is an eigenvector

\[
\|\vec{v}\| = \|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|
\]

therefore, we must have \( \lambda = \pm 1 \).

Theorem

The only possible real eigenvalues of an orthogonal matrix are 1 and \(-1\).

Flash-Forward: When a matrix as above has \( |\lambda| = 1 \), and \( \lambda \) is allowed to be complex; there are infinitely many possibilities \( \lambda = \cos \theta + i \sin \theta \). A length-preserving matrix with complex eigenvalues is usually called a Unitary Matrix; the Orthogonal Matrices are special cases of Unitary Matrices (with real eigenvalues).
Matrices with Eigenvalue 0

**Example**

By definition 0 is an eigenvalue if we can find a non-zero $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = 0\vec{x} = \vec{0}$. That means 0 is an eigenvalue of $A$ if and only if $\ker(A) \neq \{\vec{0}\}$, i.e. $A$ is non-invertible.

We add this to our list from (Notes 3.3)

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**Characteristics of Invertible Matrices**

Summary: Invertible Matrices

For an $n \times n$ matrix $A$, the following statements are equivalent:

i. $A$ is invertible

ii. The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$

iii. $\text{rref}(A) = I_n$

iv. $\text{rank}(A) = n$

v. $\text{im}(A) = \mathbb{R}^n$

vi. $\ker(A) = \{\vec{0}\}$

vii. The column vectors of $A$ form a basis of $\mathbb{R}^n$

viii. The column vectors of $A$ span $\mathbb{R}^n$

ix. The column vectors of $A$ are linearly independent

x. $\det(A) \neq 0$

xi. 0 is not an eigenvalue of $A$.

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**Suggested Problems 7.1**

**Available on Learning Glass videos:**

7.1 — 1, 3, 5, 7, 15, 17, 21

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**Lecture – Book Roadmap**

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