Outline

1. Student Learning Objectives
   - SLOs: Eigen-values and vectors: Diagonalization

2. Eigenvalues and Eigenvectors
   - Introduction
   - Baby Steps...

3. Diagonalization
   - Motivating Example
   - Definitions, etc...

4. Suggested Problems
   - Suggested Problems 7.1
   - Lecture–Book Roadmap
After this lecture you should know how

- Matrix Diagonalization
- Similarity Transformation
- Eigenvalues, Eigenvectors, and Eigenbases

are inter-related.

\[ S^{-1} \quad A \quad S \quad D \]
Figure: Consider linear transformations $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. What do we know? — They rotate, reflect, and stretch our input space (Left panel) in various ways; two examples shown in the Center and Right panels. An eigenvector, $\vec{v}$, of a linear transformation is a vector whose orientation is preserved by the transformation, i.e. $\vec{v} \parallel A\vec{v}$, i.e.

$$A\vec{v} = \lambda \vec{v},$$

and the scalar $\lambda$ is the eigenvalue associated with $\vec{v}$. 
Why Should We Care About Eigenvalues?

From a “pure” linear algebra perspective, operations on eigenvectors are easy, since they are just (multiplicative) scalings. In applications the eigenvector-eigenvalue pair describe some fundamental property of a “system” (something we are using a mathematical model to describe):

- **Vibrations**, either in strings (guitars, pianos, etc) or other structures (bridges, tall buildings): the eigenvalue is the frequency, and the eigenvector is the deformation. [Tacoma Bridge]

- In statistics, **Principal Component Analysis**, is an eigenvector-eigenvalue analysis of the correlation matrix, and is used to study large data sets, such as those encountered in data mining, chemical research, psychology, and in marketing. **Buzzwords**: “Big Data,” and “Page Rank.”
Why Should We Care About Eigenvalues?

**Figure:** “Eigenvalue Analysis of Microwave Oven.” *Haider, Siddique, Abbas, and Ahmed* — *International Journal of Scientific & Engineering Research, Volume 4, Issue 9, September-2013, pp. 2473.*
**Figure:** Space and time evolution of a three-cells hopping state found in simulations of the Kuramoto-Sivashinsky equation. The cells move non-uniformly and their shapes change periodically. Parameter values are: \( \varepsilon = 0.32, \eta_1 = 1.0, \eta_2 = 0.013, \) and \( R = 7.7475 \). — Blomgren, Gasner, and Palacios, “Hopping Behavior in the Kuramoto-Sivashinsky Equation.” Chaos: An Interdisciplinary Journal of Nonlinear Science, volume 15; March 28, 2005.
Example: Energy Modes in the Hydrogen Wave Function

OK, we may not be quite ready to identify the energy states in the hydrogen atom... yet. Let us, momentarily, retreat to our safe Linear Algebra “universe.”
Motivating Example

Given the matrices

\[ A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix} \]

ponder the “fun” of

1. computing:
   - \( A^5 \), \( \text{rank}(A) \), \( \text{det}(A) \), the basis of \( \ker(A) \), and the basis of \( \text{im}(A) \)

2. computing:
   - \( B^5 \), \( \text{rank}(B) \), \( \text{det}(B) \), the basis of \( \ker(B) \), and the basis of \( \text{im}(B) \)
Motivating Example (A)

For Matrix $A$ we can write down the answers quickly:

$$\text{rank}(A) = 3, \quad \text{det}(A) = 0$$

$$A^5 = \begin{bmatrix} (-1)^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1^5 & 0 \\ 0 & 0 & 0 & 2^5 \end{bmatrix} = \begin{bmatrix} (-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$$\ker(A) \in \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \text{im}(A) \in \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$
Motivating Example (B)

Now, for matrix $B$ it’s a “bit” more work... and maybe not immediately obvious that:

$$\text{rank}(B) = 2, \quad \text{det}(B) = 0$$

$$B^5 = \begin{bmatrix} 412,928 & 413,184 & 413,440 & 413,696 \\ 1,052,928 & 1,053,184 & 1,053,440 & 1,053,696 \\ 1,187,072 & 1,186,816 & 1,186,560 & 1,186,304 \\ 547,072 & 546,816 & 546,560 & 546,304 \end{bmatrix}$$

$$\ker(B) \in \text{span} \begin{Bmatrix} \begin{bmatrix}1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix}2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \end{Bmatrix}, \quad \text{im}(B) \in \text{span} \begin{Bmatrix} \begin{bmatrix}1 \\ 5 \\ 9 \\ 5 \end{bmatrix}, \begin{bmatrix}2 \\ 6 \\ 8 \\ 4 \end{bmatrix} \end{Bmatrix}$$

We come to the realization that diagonal matrices are our friends!
Definition (Diagonalizable Matrices)

Consider a linear transformation \( T(\vec{x}) = A\vec{x}; \ (T : \mathbb{R}^n \to \mathbb{R}^n) \).
Then \( A \) (and/or \( T \)) is said to be \textit{diagonalizable} if the matrix \( B \) of \( T \) with respect to some basis is diagonal.

By previous discussion (\textbf{NOTES 3.4}), the matrix \( A \) is diagonalizable \textit{if and only if} it is similar to some diagonal matrix \( B \); meaning that there exists some invertible matrix \( S \), so that

\[
S^{-1}AS = B \text{ is a diagonal matrix.}
\]

Definition (Diagonalization of a Matrix)

To \textit{diagonalize} a square matrix \( A \) means to find an invertible matrix \( S \) and a diagonal matrix \( B \) such that \( S^{-1}AS = B \).
Definition (Eigenvectors, Eigenvalues, and Eigenbases)

Consider a linear transformation $T(\vec{x}) = A\vec{x}; \ (T : \mathbb{R}^n \to \mathbb{R}^n)$. A non-zero vector $\vec{v} \in \mathbb{R}^n$ is called an eigenvector of $A$ (and/or $T$) if

$$A\vec{v} = \lambda\vec{v},$$

for some $\lambda \in \mathbb{R}$. This $\lambda$ is called the eigenvalue associated with the eigenvector $\vec{v}$.

A basis $\vec{v}_1, \ldots, \vec{v}_n$ of $\mathbb{R}^n$ is called an eigenbasis for $A$ (and/or $T$) if the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are eigenvectors of $A$, i.e.

$$A\vec{v}_k = \lambda_k \vec{v}_k, \quad k = 1, \ldots, n$$

for some scalars $\lambda_1, \ldots, \lambda_n$. 
Example (Repeated Multiplication by $A$)

If $\vec{v}$ is an eigenvector of $A$, then $A\vec{v} = \lambda \vec{v}$; and

$$A^2 \vec{v} = A(A\vec{v}) = A(\lambda \vec{v}) = \lambda A\vec{v} = \lambda^2 \vec{v}.$$ 

If it does not take a lot of imagination to realize

$$A^k \vec{v} = \lambda^k \vec{v}$$
Example $A^k \vec{v}$ in $\mathbb{R}^2$

Assume we have an eigenbasis \{\vec{v}_1, \vec{v}_2\}, then any $\vec{w} \in \mathbb{R}^2$ can be written as $\vec{w} = a_1 \vec{v}_1 + a_2 \vec{v}_2$ for unique scalars $a_1$ and $a_2$; now

$$A\vec{w} = A(a_1 \vec{v}_1 + a_2 \vec{v}_2) = a_1 A\vec{v}_1 + a_2 A\vec{v}_2 = a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_2 \vec{v}_2$$

and

$$A^k \vec{w} = A^{k-1}(A(a_1 \vec{v}_1 + a_2 \vec{v}_2)) = A^{k-1}(a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_2 \vec{v}_2) = a_1 \lambda_1^k \vec{v}_1 + a_2 \lambda_2^k \vec{v}_2$$
Example $A^k \vec{v}$ in $\mathbb{R}^2$

For each $A_i$, the eigenvalues are $\lambda_1 = 10/9$ and $\lambda_2 = 9/10$, and the determinant $\det(A) = 1$. The transformations are applied to vectors $\vec{v}$ in $\mathbb{R}^2$. The diagrams illustrate the effect of the transformations on the unit square.
Example $A^k \vec{v}$ in $\mathbb{R}^2$

$A^6$ applied; $\lambda_1 = 10/9$, $\lambda_2 = 9/10$; det$(A) = 1$

$A^7$ applied; $\lambda_1 = 10/9$, $\lambda_2 = 9/10$; det$(A) = 1$

$A^8$ applied; $\lambda_1 = 10/9$, $\lambda_2 = 9/10$; det$(A) = 1$

$A^9$ applied; $\lambda_1 = 10/9$, $\lambda_2 = 9/10$; det$(A) = 1$

$A^{10}$ applied; $\lambda_1 = 10/9$, $\lambda_2 = 9/10$; det$(A) = 1$

$A^0$ applied; $\lambda_1 = 10/9$, $\lambda_2 = 9/10$; det$(A) = 1$
Theorem (Eigenbases and Diagonalization)

The matrix $A$ is diagonalizable if and only if there exists an eigenbasis for $A$. If $\vec{v}_1, \ldots, \vec{v}_n$ is an eigenbasis for $A$, with $A\vec{v}_1 = \lambda_1 \vec{v}_1, \ldots, A\vec{v}_n = \lambda_n \vec{v}_n$, then the matrices

$$S = [\vec{v}_1 \: \cdots \: \vec{v}_n], \quad \text{and} \quad B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

will diagonalize $A$, meaning $S^{-1}AS = B$.

Conversely, if the matrices $S$ and $B$ diagonalize $A$, then the columns of $S$ will form an eigenbasis for $A$, and the diagonal entries of $B$ will be the associated eigenvalues.
Example \((T(\vec{x}) = I_n\vec{x} = \vec{x})\)

Find all the eigenvalues of eigenvectors of the identity matrix \(I_n\).

**Solution:** Since

\[
I_n\vec{x} = 1\vec{x} \quad \forall \vec{x} \in \mathbb{R}^n,
\]

it follows that all \(\vec{x} \in \mathbb{R}^n\) are eigenvectors of \(I_n\) with associated eigenvalues 1.

Therefore, all bases of \(\mathbb{R}^n\) are eigenbases for \(I_n\), and clearly the already diagonal matrix \(I_n\) is diagonalizable. Any invertible \(S\) will do the trick: \(S^{-1}I_nS = S^{-1}S = I_n\).
Eigenvalues of a Projection

Example (Projection: Setup)

Let \( \vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \), and consider the projection onto the line \( L = \text{span} \{ \vec{w} \} \):

\[
T(\vec{x}) = \text{proj}_L(\vec{x}) = \frac{\vec{x} \circ \vec{w}}{\|\vec{w}\|^2} \vec{w} = P\vec{x}
\]

where (NOTES 2.2)

\[
P = \frac{1}{25} \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}
\]
Example (Projection: Getting Started…)

(Restating old results in this new context —) Any vector $\parallel$ to $L$ ($\vec{w}$) will be projected onto itself (hence it is an eigenvector with eigenvalue 1); and any vector $\perp$ to $L$ ($\vec{w}$) will be projected onto $\vec{0}$, so it is an eigenvector with eigenvalue 0;

$$P\vec{x}_\parallel = 1\vec{x}_\parallel, \quad P\vec{x}_\perp = 0\vec{x}_\perp$$

one eigenbasis is

$$\mathcal{B} = (\vec{x}_\parallel, \vec{x}_\perp), \quad \text{where} \quad \vec{x}_\parallel = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \vec{x}_\perp = \begin{bmatrix} -3 \\ 4 \end{bmatrix},$$
Example (Projection: ...Moving Along)

The $B$-matrix (which expresses $T$ in the $\mathcal{B}$-basis) is

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

Recall that the first column is the coefficients, $\langle 1, 0 \rangle$ of $T(\vec{x}^\parallel)$ in the basis $\mathcal{B} = (\vec{x}^\parallel, \vec{x}^\perp)$; and the second column is the coefficients $\langle 0, 0 \rangle$ of $T(\vec{x}^\perp)$.

The matrices $B$ and

$$S = \begin{bmatrix} \vec{x}^\parallel & \vec{x}^\perp \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$$

diagonalize $P$. 
Example (Rotation by $\pi/2$ (90°))

Let $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $T(\vec{x}) = R\vec{x}$ be the rotation transformation.

Now, given any $\vec{x} \in \mathbb{R}^2$, $\vec{x}$ is not parallel to $R\vec{x}$.

As long as we insist in REAL eigenvectors and REAL eigenvalues, we find none... Complex eigenvalues will be considered soon.

Flash-Forward: The eigenvalues of the matrix above are $0 + 1i$, and $0 - 1i$; where $i = \sqrt{-1}$. 
Example (Orthogonal $A \in \mathbb{R}^{n \times n}$)

Let $A$ be an orthogonal matrix; then $T(\vec{x}) = A\vec{x}$ preserves length, so if/when $\vec{v}$ is an eigenvector

$$\| \vec{v} \| = \| A\vec{v} \| = \| \lambda \vec{v} \| = |\lambda| \| \vec{v} \|$$

therefore, we must have $\lambda = \pm 1$.

Theorem

The only possible real eigenvalues of an orthogonal matrix are 1 and $-1$.

Flash-Forward: When a matrix as above has $|\lambda| = 1$, and $\lambda$ is allowed to be complex; there are infinitely many possibilities $\lambda = \cos \theta + i \sin \theta$. A length-preserving matrix with complex eigenvalues is usually called a Unitary Matrix; the Orthogonal Matrices are special cases of Unitary Matrices (with real eigenvalues).
Matrices with Eigenvalue 0

Example
By definition 0 is an eigenvalue if we can find a non-zero $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = 0\vec{x} = \vec{0}$. That means 0 is an eigenvalue of $A$ if and only if $\ker(A) \neq \{\vec{0}\}$, i.e. $A$ is non-invertible.

We add this to our list from (NOTES 3.3)
Characteristics of Invertible Matrices

Summary: Invertible Matrices

For an $n \times n$ matrix $A$, the following statements are equivalent:

i. $A$ is invertible

ii. The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$

iii. $\text{rref}(A) = I_n$

iv. $\text{rank}(A) = n$

v. $\text{im}(A) = \mathbb{R}^n$

vi. $\text{ker}(A) = \{\vec{0}\}$

vii. The column vectors of $A$ form a basis of $\mathbb{R}^n$

viii. The column vectors of $A$ span $\mathbb{R}^n$

ix. The column vectors of $A$ are linearly independent

x. $\det(A) \neq 0.$

xi. $0$ is not an eigenvalue of $A.$
Available on Learning Glass videos:
7.1 — 1, 3, 5, 7, 15, 17, 21
### Lecture – Book Roadmap

<table>
<thead>
<tr>
<th>Lecture</th>
<th>Book, ([GS5-])</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1</td>
<td>§6.1</td>
</tr>
<tr>
<td>7.2</td>
<td>§6.1, §6.2</td>
</tr>
<tr>
<td>7.3</td>
<td>§6.1, §6.2</td>
</tr>
<tr>
<td>7.5</td>
<td>§6.1, §6.2</td>
</tr>
</tbody>
</table>