Outline

1 Approximation Theory
   • Fundamentals
   • 2D: Piecewise Linear
   • Putting the Theorem to Use
Let $V$ be a Hilbert space with scalar product $(\circ, \circ)_V$ and corresponding norm $||\circ||_V$. Suppose that $a(\circ, \circ)$ is a bilinear form on $V \times V$ and $L$ is a linear form on $V$ such that the following criteria are met:

1. $a(\circ, \circ)$ is symmetric.
2. $a(\circ, \circ)$ is continuous, i.e. there is a constant $\gamma > 0$ such that $|a(v, w)| \leq \gamma ||v||_V ||w||_V \ \forall v, w \in V$.
3. $a(\circ, \circ)$ is $V$-elliptic, i.e. there is a constant $\alpha > 0$ such that $a(v, v) > \alpha ||v||^2_V \ \forall v \in V$.
4. $L$ is continuous i.e. there is a constant $\Lambda > 0$ such that $|L(v)| \leq \Lambda ||v||_V, \ \forall v \in V$. 

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For an elliptic problem satisfying conditions [1]–[4] we have the following error estimate for the FEM solution

$$\| u - u_h \|_V \leq \frac{\gamma}{\alpha} \| u - v \|_V, \quad \forall v \in V_h$$

In particular, we can choose $v = \pi_h u \in V_h$ to be a suitable interpolant of $u$, and by bounding the interpolation error $\| u - \pi_h u \|_V$ we obtain an error estimate of the error $\| u - u_h \|_V$.

We will now study the problem of estimating the interpolation error. The interpolant is usually chosen so that the degrees of freedom for $V_h$ agree for $u$ and $u_h$. Thus it is sufficient to estimate the error locally on each element $t \in T_h$. 
Interpolation with Piecewise Linear Functions in 2D

We consider the familiar case, $\Omega \subset \mathbb{R}^2$

$$V = H^1(\Omega), \quad V_h = \{ v \in H^1(\Omega) : v|_t \in P_1(t), \ \forall t \in T_h \}$$

For $t \in T_h$ we define

$$h_t = \text{the diameter of } t \text{ (the longest side of } t)$$
$$\rho_t = \text{the diameter of the circle inscribed in } t$$
$$h = \max_{t \in T_h} h_t$$
We are interested in a family of triangulations \( \{ T_h \} \) indexed by \( h \). Further, we will assume that there is a constant \( \beta > 0 \) such that

\[
\frac{\rho_t}{h_t} \geq \beta
\]

This is a restriction on how “long and narrow” the triangles are allowed to get.

Let \( \{ n_i \}_{i=1}^N \) be the enumeration of the nodes of \( T_h \). Given \( u \in C^0(\Omega) \) we define the interpolant \( \pi_h u \in V_h \) by

\[
\pi_h u(n_i) = u(n_i), \quad i = 1, 2, \ldots, N
\]

Thus \( \pi_h u \) is the piecewise linear function agreeing with \( u \) at the nodes of \( T_h \).
Quality Control: Theorem #1

Theorem (#1)

Let \( t \in T_h \) be a triangle with vertices \( a^i \), \( i = 1, 2, 3 \). Given \( v \in C^0(t) \), let the interpolant \( \pi v \in P_1(t) \) be defined by

\[
\pi v(a^i) = v(a^i), \quad i = 1, 2, 3
\]

Then

\[
\| v - \pi v \|_{L_\infty(t)} \leq 2h_t^2 \max_{|\alpha| = 2} \| D^\alpha v \|_{L_\infty(t)},
\]

\[
\max_{|\alpha| = 1} \| D^\alpha (v - \pi v) \|_{L_\infty(t)} \leq 6 \frac{h_t^2}{\rho_t} \max_{|\alpha| = 2} \| D^\alpha v \|_{L_\infty(t)}
\]

where

\[
\| v \|_{L_\infty(t)} = \max_{\tilde{x} \in t} |v(\tilde{x})|.
\]
Note that the errors $[v - \pi v]$ and $[D^\alpha (v - \pi v)]$ both depend on the second partial derivatives of $v$. — The second derivatives measure the curvature of the surface.

**Proof:** Let $\lambda_i(\tilde{x})$, $i = 1, 2, 3$ be the basis functions for $P_1(t)$. A general function $w \in P_1(t)$ has the following representation:

$$w(\tilde{x}) = \sum_{i=1}^{3} w(a^i)\lambda_i(\tilde{x}), \quad \tilde{x} \in t$$

so that in particular

$$\pi v(\tilde{x}) = \sum_{i=1}^{3} v(a^i)\lambda_i(\tilde{x}), \quad \tilde{x} \in t$$
Using Taylor expansions we derive a representation for the errors:

\[ v(\tilde{y}) = v(\tilde{x}) + \sum_{j=1}^{2} \left[ \frac{\partial v(\tilde{x})}{\partial x_j} (y_j - x_j) \right] + R(x, y) \]

where the remainder term is

\[ R(\tilde{x}, \tilde{y}) = \frac{1}{2} \sum_{i,j=1}^{2} \left[ \frac{\partial^2 v(\tilde{\xi})}{\partial x_i \partial x_j} (y_i - x_i)(y_j - x_j) \right] \]

with \( \tilde{\xi} \) a point on the line segment connection \( \tilde{x} \) and \( \tilde{y} \).
In particular we can let $\tilde{y} - a^i$, and we get

$$v(a^i) = v(\tilde{x}) + p_i(\tilde{x}) + R_i(\tilde{x})$$

where

$$p_i(\tilde{x}) = \sum_{j=1}^{2} \left[ \frac{\partial v(\tilde{x})}{\partial x_j} (a^i_j - x_j) \right]$$

$$R_i(\tilde{x}) = R(\tilde{x}, a^i)$$

Since, by design (definition)

$$|a^i_j - x_j| \leq h_t, \quad i = 1, 2, 3, \quad j = 1, 2$$

we have the following estimate of the remainder term $R(\tilde{x})$:

$$R_i(\tilde{x}) \leq 2h_t^2 \max_{|\alpha|=2} \| D^\alpha v \|_{L_\infty(t)}, \quad i = 1, 2, 3$$
Writing down the interpolating polynomial in terms of the Taylor expansion:

\[ \pi v(\tilde{x}) = \sum_{i=1}^{3} v(a^i) \lambda_i(\tilde{x}) = \sum_{i=1}^{3} [(v(\tilde{x}) + p_i(\tilde{x}) + R_i(\tilde{x})) \lambda_i(\tilde{x})] \]

\[ = v(\tilde{x}) \sum_{i=1}^{3} \lambda_i(\tilde{x}) + \sum_{i=1}^{3} p_i(\tilde{x}) \lambda_i(\tilde{x}) + \sum_{i=1}^{3} R_i(\tilde{x}) \lambda_i(\tilde{x}) \]
In order to proceed, we need to following lemma:

Lemma (#1)

For \( j = 1, 2 \) and \( \tilde{x} \in t \) we have

\[
\begin{align*}
(L-1) \sum_{i=1}^{3} \lambda_i(\tilde{x}) &= 1, \\
(L-2) \sum_{i=1}^{3} p_i(\tilde{x})\lambda_i(\tilde{x}) &= 0 \\
(L-3) \sum_{i=1}^{3} \left[ \frac{\partial}{\partial x_j} \lambda_i(\tilde{x}) \right] &= 0, \\
(L-4) \sum_{i=1}^{3} \left[ p_i(\tilde{x})\frac{\partial \lambda_i(\tilde{x})}{\partial x_j} \right] &= \frac{\partial v(\tilde{x})}{\partial x_j}
\end{align*}
\]
Proof of Lemma

**Proof:** The proof is based on the observation that $\pi v \equiv v$, $\forall v \in P_1(t)$, which follows from the uniqueness of functions in $P_1(t)$ with prescribed values at the nodes.

If we choose $v(\tilde{x}) = 1$, then we get

$$1 = \sum_{i=1}^{3} \lambda_i(\tilde{x})$$

and in this case $p_1(\tilde{x}) = R_1(\tilde{x}) = 0$. This proves (L-1) and (L-3) directly.

To show (L-2) we choose $v(\tilde{x}) = d_1x_1 + d_2x_2$ in

$$\pi v(\tilde{x}) = v(\tilde{x}) \sum_{i=1}^{3} \lambda_i(\tilde{x}) + \sum_{i=1}^{3} p_i(\tilde{x})\lambda_i(\tilde{x}) + \sum_{i=1}^{3} R_i(\tilde{x})\lambda_i(\tilde{x})$$
Proof of Lemma

With this linear choice of $\nu(\tilde{x})$, we have $\pi \nu = \nu$, and further

$$p_i(\tilde{x}) = d_1(a^i_1 - x_1) + d_2(a^i_2 - x_2), \quad R_i(\tilde{x}) \equiv 0$$

Hence,

$$\nu(\tilde{x}) = \nu(\tilde{x}) + \sum_{i=1}^{3} \left[ d_1(a^i_1 - x_1) + d_2(a^i_2 - x_2) \right] \lambda_i(\tilde{x})$$

It follows that

$$\sum_{i=1}^{3} \left[ d_1(a^i_1 - x_1) + d_2(a^i_2 - x_2) \right] \lambda_i(\tilde{x}) = 0, \quad \forall d_1, d_2 \in \mathbb{R}$$

This proves $(L-2)$ by choosing $d_i = \frac{\partial \nu(\tilde{x})}{\partial x_i}$, $i = 1, 2$. The proof for $(L-4)$ is similar.
Back to the Proof of the Theorem

We now apply (L-1) and (L-2) to

\[ \pi v(\tilde{x}) = v(\tilde{x}) \sum_{i=1}^{3} \lambda_i(\tilde{x}) + \sum_{i=1}^{3} p_i(\tilde{x}) \lambda_i(\tilde{x}) + \sum_{i=1}^{3} R_i(\tilde{x}) \lambda_i(\tilde{x}) \]

and get

\[ \pi v(\tilde{x}) = v(\tilde{x}) + \sum_{i=1}^{3} R_i(\tilde{x}) \lambda_i(\tilde{x}) \]

Hence the interpolation error can be written:

\[ v(\tilde{x}) - \pi v(\tilde{x}) = - \sum_{i=1}^{3} R_i(\tilde{x}) \lambda_i(\tilde{x}) \]
Proof: End of Part-1, Start of Part-2

We can use our previous estimate of the remainder term to get

\[
|v(\tilde{x}) - v(\tilde{x})| \leq \sum_{i=1}^{3} |R_i(\tilde{x})| \lambda_i(\tilde{x}) \\
\leq \max_i |R_i(\tilde{x})| \sum_{i=1}^{3} \lambda_i(\tilde{x}) \\
\leq 2h_t^2 \max_{|\alpha|=2} \|D^\alpha v\|_{L_\infty(t)}, \quad i = 1, 2, 3
\]

This shows the first part of the theorem. △

To prove the second part, we differentiate

\[
\pi v(\tilde{x}) = \sum_{i=1}^{3} v(a^i) \lambda_i(\tilde{x}), \quad \tilde{x} \in t
\]

with respect to \(x_1\) and get

\[
\frac{\partial \pi v(\tilde{x})}{\partial x_j} = \sum_{i=1}^{3} v(a^i) \frac{\partial \lambda_i(\tilde{x})}{\partial x_j}, \quad \tilde{x} \in t
\]
We again use
\[ v(a^i) = v(\tilde{x}) + p_i(\tilde{x}) + R_i(\tilde{x}) \]
which gives, using (L-3) and (L-4)
\[
\frac{\partial \pi v(\tilde{x})}{\partial x_j} = \frac{\partial v(\tilde{x})}{\partial x_j} + \sum_{i=1}^{3} R_i(\tilde{x}) \frac{\partial \lambda_i(\tilde{x})}{\partial x_j}
\]
It is now easy to see that
\[
\max_{\tilde{x} \in t} \left| \frac{\partial \lambda_i(\tilde{x})}{\partial x_j} \right| \leq \frac{1}{\rho_t}
\]
Combining this with the estimate for \( R_i(\tilde{x}) \) now finally gives
\[
\left| \frac{\partial v(\tilde{x})}{\partial x_j} - \frac{\partial \pi v(\tilde{x})}{\partial x_j} \right| \leq 6 \frac{h_t^2}{\rho_t \max_{|\alpha|=2} \| D^\alpha v(\tilde{x}) \|_{L_\infty(t)}}
\]
Let \( j = 1, 2 \) and the proof is complete. \( \square \)
Making the Theorem More Useful

Since the theorem gives the estimate of the interpolation error using the $L_\infty(t)$-norm it is not well suited to give estimates for $\| u - u_h \|_{H^1(\Omega)}$ (which we care about) involving the $L_2$-norm.

We state an analogue of Theorem #1 using $H^r(\Omega)$-semi-norms:

$$|v|_{H^r(\Omega)} = \sqrt{\sum_{|\alpha|=r} \int_{\Omega} |D^\alpha v|^2 d\tilde{x}}$$

Note that $|v|_{H^r(\Omega)}$ measures the $L_2(\Omega)$-norm of the partial derivatives of $v$ of order exactly $r$.

$|v|_{H^r(\Omega)}$ is not a norm: Let $r = 1$ and $v(\tilde{x}) = 1$, then $|v|_{H^r(\Omega)} = 0$ even though $v \not\equiv 0$. 
Theorem #2

Theorem (#2)

Let \( t \in T_h \) be a triangle with vertices \( a^i, i = 1, 2, 3 \). Given \( v \in C^0(t) \), let the interpolant \( \pi v \in P_1(t) \) be defined by

\[
\pi v(a^i) = v(a^i), \quad i = 1, 2, 3
\]

Then

\[
\| v - \pi v \|_{L^2(t)} \leq C h_t^2 |v|_{H^2(t)}
\]

\[
| v - \pi v |_{H^1(t)} \leq C \frac{h_t^2}{\rho_t} |v|_{H^2(\Omega)}
\]

Theorem #2 has the same structure as theorem #1. The difference is the [semi]norm involved. In theorem #1 we measured everything in terms of the \( L_\infty \)-norm, and here we use the \( L_2 \)-norm.

The proof for theorem #2 is more complicated than the one for theorem #1 (technical complications).
Application of Theorem #2

We now apply Theorem #2 to estimate the **global interpolation errors** \( \| u - \pi_h u \|_{L^2(\Omega)} \) and \( | u - \pi_h u |_{H^1(\Omega)} \).

\[
\| u - \pi_h u \|_{L^2(\Omega)}^2 = \sum_{t \in T_h} \| u - \pi_h u \|_{L^2(t)}^2 
\leq \sum_{t \in T_h} C^2 h_t^4 |u|_{H^2(t)}^2 
\leq C^2 h_t^4 \sum_{t \in T_h} |u|_{H^2(t)}^2 
= C^2 h_t^4 |u|_{H^2(\Omega)}^2
\]

Hence

\[
\| u - \pi_h u \|_{L^2(\Omega)} \leq Ch_t^2 |u|_{H^2(\Omega)}
\]
Similarly, using \( \frac{h_t}{\rho_t} \leq \frac{1}{\beta} \)

\[
|u - \pi_h u|_{H^2(\Omega)}^2 \leq \sum_{t \in T_h} C^2 \frac{h_t^4}{\rho_t^2} |u|_{H^2(t)}^2
\]

\[
\leq \sum_{t \in T_h} C^2 \frac{h_t^2}{\beta^2} |u|_{H^2(t)}^2
\]

\[
\leq C^2 \frac{h^2}{\beta^2} |u|_{H^2(\Omega)}^2
\]

Hence,

\[
|u - \pi_h u|_{H^2(\Omega)} \leq \frac{Ch}{\beta} |u|_{H^2(\Omega)}
\]
Interpolation with Polynomials of Higher Degree

If we work with piecewise polynomials of degree \( r \geq 1 \), we typically have estimates of the form:

\[
\| u - \pi_h u \|_{L^2(\Omega)} \leq C h^{r+1} |u|_{H^{r+1}(\Omega)}
\]

\[
| u - \pi_h u |_{H^1(\Omega)} \leq C h^r |u|_{H^{r+1}(\Omega)}
\]

If \( V_h \subset H^2(\Omega) \) then we also have

\[
| u - \pi_h u |_{H^2(\Omega)} \leq C h^{r-1} |u|_{H^{r+1}(\Omega)}
\]

If \( V_h \subset H^3(\Omega) \) then we also have

\[
| u - \pi_h u |_{H^3(\Omega)} \leq C h^{r-2} |u|_{H^{r+1}(\Omega)}
\]
Example: $P_5(t)$-Elements

With $V_h = \{ c \in C^1(\Omega) : \nu|_t \in P_5(t), \forall t \in T_h \}$ and for $\nu \in C^2(\Omega)$, the interpolant matches all the parameters:

\[
\begin{align*}
\| u - \pi_h u \|_{L^2(\Omega)} & \leq C h^6 | u |_{H^6(\Omega)} \\
| u - \pi_h u |_{H^1(\Omega)} & \leq C h^5 | u |_{H^6(\Omega)} \\
| u - \pi_h u |_{H^2(\Omega)} & \leq C h^4 | u |_{H^6(\Omega)}
\end{align*}
\]