Numerical Solutions to PDEs
Lecture Notes #1 — Introduction

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Outline

1. The Professor
   - Academic Life
   - Contact Information, Office Hours
   - Non-Academic Life

2. The Class — Overview
   - Literature & Syllabus
   - Grading
   - Expectations and Procedures

3. The Class...
   - Resources
   - Formal Prerequisites

4. Introduction
   - Some PDE Models

Academic Life

MSc

- MSc. Engineering Physics, Royal Institute of Technology (KTH), Stockholm, Sweden. Thesis Advisers: Michael Benedicks, Department of Mathematics KTH, and Erik Aurell, Stockholm University, Department of Mathematics. Thesis Topic: “A Renormalization Technique for Families with Flat Maxima.”

PhD


Figure: Bifurcation diagram for the family $f_{2,1/2}$ [Blomgren-1994]

Figure: The noisy (SNR = 4.62 dB), and recovered space curves. Notice how the edges are recovered. [Blomgren-1998]
Research Associate. Stanford University, Department of Mathematics. Main Focus: Time Reversal and Imaging in Random Media (with George Papanicolaou, et al.)

Figure: Comparison of the theoretical formula for a medium with $L = 600 \text{ m}$, $a_e = 195 \text{ m}$, $\gamma = 2.12 \times 10^{-5} \text{ m}^{-1}$. [LEFT] shows a homogeneous medium, $\gamma = 0$, with $a = 40 \text{ m}$ TRM (in red / wide Fresnel zone), and a random medium with $\gamma = 2.12 \times 10^{-5}$ (in blue). [RIGHT] shows $\gamma = 0$, with $a = a_e = 195 \text{ m}$ (in red), and $\gamma = 2.12 \times 10^{-5}$, with $a = 40 \text{ m}$ (in blue). The match confirms the validity of [the theory]. [Blomgren-Papanicolaou-Zhao-2002]

Fun Times... ⇒ Endurance Sports

- **Triathlons:**
  - (13) Ironman distance (2.4 + 112 + 26.2) — 11:48:57 [PR]
  - (16) Half Ironman distance — 5:14:20

- **Running**
  - (1) 100k Race (62.1 miles) — 15:37:46
  - (1) Trail Double-marathon (52 miles) — 10:59:00
  - (5) Trail 50-mile races — 9:08:46
  - (8) Trail 50k (31 mile) races — 5:20:57
  - (15) Road/Trail Marathons — 3:26:19 (7:52/mi)
  - (28) Road/Trail Half Marathons — 1:36:25 (7:21/mi)
Math 693b: Literature


Author: John C. Strikwerda.

Publisher: Society for Industrial and Applied Mathematics.


Class notes and web-page.

Syllabus

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Hyperbolic Partial Differential Equations</td>
</tr>
<tr>
<td>2</td>
<td>Analysis of Finite Difference Schemes</td>
</tr>
<tr>
<td>3</td>
<td>Order of Accuracy of Finite Difference Schemes</td>
</tr>
<tr>
<td>4</td>
<td>Stability for Multistep Methods</td>
</tr>
<tr>
<td>5</td>
<td>Dissipation and Dispersion</td>
</tr>
<tr>
<td>6</td>
<td>Parabolic Partial Differential Equations</td>
</tr>
<tr>
<td>7</td>
<td>Systems of PDEs in Higher Dimensions</td>
</tr>
<tr>
<td>8</td>
<td>Second-Order Equations</td>
</tr>
<tr>
<td>9</td>
<td>Analysis of Well-Posed and Stable Problems</td>
</tr>
<tr>
<td>10</td>
<td>Convergence Estimates for Initial Value Problems</td>
</tr>
<tr>
<td>11</td>
<td>Well-Posed and Stable Initial-Boundary Value Problems</td>
</tr>
<tr>
<td>12</td>
<td>Elliptic Partial Differential Equations and Difference Schemes</td>
</tr>
<tr>
<td>13</td>
<td>Linear Iterative Methods</td>
</tr>
<tr>
<td>14</td>
<td>Steepest Descent and Conjugate Gradient Methods</td>
</tr>
</tbody>
</table>

Grading

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>Homework*</td>
<td>50%</td>
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<tr>
<td>Project×</td>
<td>50%</td>
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</tbody>
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* ≈ 7 assignments; first ≈ 3 mostly theoretical with some computational components, last ≈ 4 “purely” computational.

× Details to be discussed.

Expectations and Procedures, I

- Most class attendance is “OPTIONAL” — Homework and announcements will be posted on the class web page. If/when you attend class:
  - Please be on time.
  - Please pay attention.
  - Please turn off mobile phones.
  - Please be courteous to other students and the instructor.
  - Abide by university statutes, and all applicable local, state, and federal laws.
Please, turn in assignments on time. (The instructor reserves the right not to accept late assignments.)

The instructor will make special arrangements for students with documented learning disabilities and will try to make accommodations for other unforeseen circumstances, e.g., illness, personal/family crises, etc. in a way that is fair to all students enrolled in the class. Please contact the instructor EARLY regarding special circumstances.

Students are expected **and encouraged** to ask questions in class!

Students are expected **and encouraged** to make use of office hours! If you cannot make it to the scheduled office hours: contact the instructor to schedule an appointment!

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**Late HW Policy**

- 10% loss of value / full week late. Examples:
  - 6 days 23 hrs 59 minutes 59 seconds late = FULL VALUE
  - 7 days 0 hrs 0 min 1 sec late = 10% LOSS OF VALUE
  - 14 days 0 hrs 0 min 1 sec late = 20% LOSS OF VALUE

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**Honesty Pledge**

The following **Honesty Pledge** must be included in all programs you submit (as part of homework and/or projects):

- I, (your name), pledge that this program is completely my own work, and that I did not take, borrow or steal code from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my code. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

**Work missing the honesty pledge may not be graded!**
Larger reports must contain the following text:

- I, (your name), pledge that this report is completely my own work, and that I did not take, borrow or steal any portions from any other person. Any and all references I used are clearly cited in the text. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies. Your signature.

- Work missing the honesty pledge may not be graded!

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**Math 693b: Computer Resources**

You need access to a computing environment in which to write your code; — you may want to use a combination of Matlab (for quick prototyping and short homework assignments) and C/C++ or Fortran (or something completely different).

Free C/C++ (gcc) and Fortran (f77) compilers are available for Linux/UNIX.

You may also want to consider buying the student version of Matlab: http://www.mathworks.com/

**SDSU students can download a copy of matlab from http://edoras.sdsu.edu/~download/matlab.html**

[License subject to change with minimal notice]

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**Math 693b: Formal Prerequisites**

**Math 531, Math 537 and Math 693a**

531 ⇒ **PDEs**
- Boundary value problems for the heat and wave equations: eigenfunction expansions, Sturm-Liouville theory and Fourier series. D’Alembert’s solution to wave equation; characteristics. Laplace’s equation, maximum principles, Bessel functions.

537 ⇒ **ODEs**
- Theory of ODEs; existence and uniqueness, dependence on initial conditions and parameters, linear systems, stability and asymptotic behavior, plane autonomous systems, series solutions at regular singular points.

693a ⇒ **Advanced Numerical Analysis (Numerical Optimization)**

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**Math 693b: Informal Prerequisites**

Math 531 and (Math 541 or Math 542 or Math 543 or Math 693a) and Mathematical Software (e.g. matlab)

Essential knowledge of PDEs, some experience with “mathematical programming” in some language (e.g. matlab), and linear algebra.

Knowledge of Fourier, Real, and Complex analysis is not required, but incredibly useful!

**If you don’t know how to write code, this class will be VERY PAINFUL.**
This class will primarily focus on **Finite Difference Methods**. We will spend some time discussing **Finite Element Methods** and/or **Mimetic Finite Difference Methods**, and possibly **Spectral Methods** in the latter part of the semester.


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### Some PDE Models

1. **The Heat Equation**

   \[ T_t - \kappa(\bar{x}) \nabla^2 T = f(\bar{x}, t) \]

   The heat equation describes heat transfer in a medium. \( \kappa \) is the thermal diffusivity and \( T \) the temperature. (heat.mpg, hmovie2d-ic6.avi)

2. **The Wave Equation**

   \[ \frac{1}{c(\bar{x})^2} \Phi_{tt} - \nabla^2 \Phi = f(\bar{x}, t) \]

   The wave equation describes propagation of waves with (location-dependent) speed \( c(\bar{x}) \). (wmovie2d-ic3.avi)

3. **The Schrödinger Equation**

   \[ i\hbar \frac{\partial \Psi(x, t)}{\partial t} - \left[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V(x) \right] \Psi(x, t) = 0 \]

   The Schrödinger equation (here in time-dependent one-dimensional form) is the fundamental equation of physics for describing **quantum mechanical behavior**. It is also often called the Schrödinger wave equation, and is a partial differential equation that describes how the wavefunction \( [\Psi(x, t)] \) of a physical system evolves over time.

   \( \hbar \) is Planck's constant divided by \( 2\pi \), \( V(x) \) a potential, and \( i \) the imaginary unit \( \sqrt{-1} \). (smovie-fn7.avi)

4. **The Korteweg-de Vries Equation**

   \[ \frac{\partial \eta(x, t)}{\partial t} = \frac{3}{2} \sqrt{\frac{g}{h}} \left( \eta(x, t) \frac{\partial \eta(x, t)}{\partial x} + \frac{2}{3} \frac{\partial \eta(x, t)}{\partial x} + \frac{1}{3} \sigma \frac{\partial^3 \eta(x, t)}{\partial x^3} \right) \]

   with \( \sigma = h^3/3 - T \hbar/(g \rho) \).

   The Korteweg-de Vries equation governs **weakly nonlinear shallow waves**. \( h \) is the channel height, \( T \) is the gravitational acceleration and \( \rho \) the density.

   The more commonly seen nondimensionalized version of the KdV equation takes the form

   \[ u_t + uu_x - 6u_{xxx} = 0 \]

   (kdv2soliton.mpg, kdv-sin.mpg)
The Fokker-Planck Equation

\[
\frac{\partial}{\partial t} G(\bar{r}, \bar{v}; \bar{r}_0, \bar{v}_0; t) + \bar{v} \cdot \nabla \bar{r} G(\bar{r}, \bar{v}; \bar{r}_0, \bar{v}_0; t) = \\
\nabla \bar{v} \cdot \xi \bar{v} G(\bar{r}, \bar{v}; \bar{r}_0, \bar{v}_0; t) + \nabla^2 \frac{\xi k T}{m} G(\bar{r}, \bar{v}; \bar{r}_0, \bar{v}_0; t)
\]

The Fokker-Planck equation describes stochastic evolution, describing drift and diffusion of a density function. \( G \) is the probability density; \( \bar{r} \) and \( \bar{r}_0 \) positions; \( \bar{v} \) and \( \bar{v}_0 \) velocities.

The Kuramoto-Sivashinsky Equation

\[
u_t + \nabla^4 u + \nabla^2 u + \frac{1}{2} |\nabla u|^2 = 0
\]

The Kuramoto-Sivashinsky equation describes the pattern formation of cellular flames stabilized on a circular porous plug burner.

Questions, Comments, Administrative Stuff...

1/30 Last day to add/drop classes; Last day to add classes; or change grading basis. No schedule adjustments allowed after 11:59 p.m. on this date.

1/30 Last day to file application for bachelors degree or advanced degree for May and August 2018 graduation.

3/23 Final day for submitting thesis (without risk) to Montezuma Publishing for thesis review to ensure graduation in May 2018.

Questions?