Recap
Previously...
Classification of PDEs: — Hyperbolic, Parabolic, and Elliptic.
Exact solutions of the hyperbolic One-Way Wave Equation
\[ u_t + a(t, x)u_x + b(t, x)u = f(t, x) \]
constant/variable coefficient, lower-order term, forcing.
Systems of hyperbolic PDEs: propagation along characteristics, initial and boundary values.
Introduction to Finite Difference Schemes: gridding; forward, backward, and central differences; notation \( v_{m+1}^{n+1} = (1 + a\lambda)v_{m}^{n} - a\lambda v_{m+1}^{n} \); Numerical example — the leapfrog scheme (dependence on \( \lambda \)).
Convergence — Definition

Definition (Convergent Scheme)
A one-step finite difference scheme approximating a PDE is a convergent scheme if for any solution to the PDE, $u(t,x)$, and solutions to the finite difference scheme, $v^n_m$, such that $v^n_m$ converges to the initial condition $u_0(x)$ as $m \cdot h$ converges to $x$, then $v^n_m$ converges to $u(t,x)$ as $(n \cdot k, m \cdot h)$ converges to $(t,x)$ as $k, h$ converge to 0.

This definition is formally not quite complete until we clarify the convergence of $v^n_m$ (on the discrete grid) to $u(t,x)$, which is continuously varying.

If this was an “analysis for PDEs” course, we would go to the trouble of filling all theoretical gaps; but, alas, we take a more practical view and leave such “minor” details as recommended homework for dark and stormy nights.

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Two Schemes: Leapfrog and Lax-Friedrichs

$$\frac{v^{n+1}_m - v^{n-1}_m}{2k} + a \frac{v^{n+1}_m - v^{n-1}_m}{2h} = 0$$  Leapfrog

$$\frac{v^{n+1}_m - \frac{1}{2}(v^{n+1}_{m+1} + v^{n-1}_{m-1})}{k} + a \frac{v^{n+1}_m - v^{n-1}_m}{2h} = 0$$  Lax-Friedrichs

Note that the Lax-Friedrichs scheme is a one-step scheme, whereas the leapfrog scheme is a 2-step scheme.

For $n$-step schemes, the definition must be changed to allow for initialization of the first $n$ time-levels. — Before we can apply an $n$-step scheme we must define $v^0_m, \ldots, v^{n-1}_m, \forall m$.

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Introduction
Examples: Leapfrog and Lax-Friedrichs Schemes
The Road To Convergence

Grid + Initialization Suitable for 1-Step Schemes

Figure: Once the first level $v^0_m$ is defined using the initial conditions $v^0_m = u_0(x_m)$ on the red grid-points, a one-step scheme, such as Lax-Friedrichs

$$\frac{v^{n+1}_m - \frac{1}{2}(v^{n+1}_{m+1} + v^{n-1}_{m-1})}{k} + a \frac{v^{n+1}_m - v^{n-1}_m}{2h} = 0$$

can be applied to compute the values $v^n_m$ at the remaining grid-points, one level at a time.

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Introduction
Examples: Leapfrog and Lax-Friedrichs Schemes
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Grid + Initialization Suitable for 2-Step Schemes

Figure: Once the first two levels $v^0_m$ and $v^1_m$ are defined using the initial conditions $v^0_m = u_0(x_m)$ and some other clever initialization scheme for $v^0_m$, on the red grid-points, a two-step scheme, such as the Leapfrog scheme

$$\frac{v^{n+1}_m - v^{n-1}_m}{2k} + a \frac{v^{n+1}_m - v^{n-1}_m}{2h} = 0$$

can be applied to compute the values $v^n_m$ at the remaining grid-points, one level at a time.

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The Lax-Friedrichs Solution of the One-Way Wave Equation

\[ u_t + u_x = 0, \quad u(0) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \]

as \( k, h \to 0 \), the solution seems to approach the exact solution.

**Consistency: Notes**

- For some schemes, we may have to restrict how \( k, h \to 0 \) in order to be consistent.
- A “smooth function” is a function which is differentiable (at least) as many times as required for the expression to make sense.
- The difference operator \( P_{k,h} \) when applied to a function of \((t,x)\) does not need to be restricted to grid-points. A forward-space difference operator applied on \( \Phi \) at \((t,x)\) gives

  \[
  \Phi(t,x+h) - \Phi(t,x) \quad \frac{h}{n+1} = \frac{1}{2}(\Phi_{m+1} + \Phi_{m-1}) + \frac{\Phi_{m+1} - \Phi_{m-1}}{2h}.
  \]

We use **Taylor expansion** around the point \((t_n,x_m)\)

\[
\begin{align*}
\Phi_{m+1} &= \Phi_m + h\Phi_x + \frac{1}{2}h^2\Phi_{xx} + \frac{1}{6}h^3\Phi_{xxx} + \mathcal{O}(h^4) \\
\Phi_{m+1} &= \Phi_m + k\Phi_t + \frac{1}{2}k^2\Phi_{tt} + \frac{1}{6}k^3\Phi_{ttt} + \mathcal{O}(k^4)
\end{align*}
\]

Note that

\[
\begin{align*}
\frac{1}{2}(\Phi_{m+1} + \Phi_{m-1}) &= \Phi_m + \frac{1}{2}h^2\Phi_{xx} + \mathcal{O}(h^4) \\
\frac{\Phi_{m+1} - \Phi_{m-1}}{2h} &= \Phi_x + \frac{1}{6}h^2\Phi_{xxx} + \mathcal{O}(h^4)
\end{align*}
\]
Checking Consistency: Lax-Friedrichs 2 of 2

We can now write

\[ P_{k,h}^{LF} \Phi = \left[ \Phi_t + a \Phi_x \right] + \frac{1}{2} k \Phi_{tt} + \frac{1}{2} h^2 \Phi_{xx} + \frac{1}{6} a h^2 \Phi_{xxx} + O \left( h^4 + \frac{h^4}{k} + k^2 \right) \]

Hence,

\[ P \Phi - P_{k,h}^{LF} \Phi = -\frac{1}{2} k \Phi_{tt} + \frac{1}{2} h^2 \Phi_{xx} - \frac{1}{6} a h^2 \Phi_{xxx} + O \left( h^4 + \frac{h^4}{k} + k^2 \right) \]

As long as \( k, h \to 0 \) in such a way that also \( \frac{h^2}{k} \to 0 \), we have \( P \Phi - P_{k,h}^{LF} \Phi \to 0 \), i.e. the Lax-Friedrichs scheme is consistent. □

Example: Consistency ≠ Convergence

We consider the one-way wave equation with constant \( a = 1 \) propagation speed, and apply the forward-space-forward-time scheme

\[ \frac{v_{m}^{n+1} - v_{m}^{n}}{k} + \frac{v_{m+1}^{n} - v_{m}^{n}}{h} = 0. \]

A quick Taylor expansion shows that this indeed is consistent with the PDE, with an error term

\[ P \Phi - P_{k,h} \Phi \sim k \Phi_{tt} + h \Phi_{xx} + O \left( k^2 + h^2 \right). \]

We rewrite the scheme using \( \lambda = k/h \):

\[ v_{m}^{n+1} = v_{m}^{n} - \frac{k}{h} \left( v_{m+1}^{n} - v_{m}^{n} \right) = (1 + \lambda) v_{m}^{n} - \lambda v_{m+1}^{n}. \]

Consistency ≠ Convergence

- Consistency implies that the solution of the PDE, if it is smooth, is an approximate solution of the finite difference scheme (FDS).
- Convergence means that a solution of the FDS approximates a solution of the PDE.
- It turns out that consistency is necessary, but not sufficient for a FDS to be convergent.

We illustrate this with an example.

Let the initial condition be given by

\[ u_0(x) = \begin{cases} 1 & \text{if } -1 \leq x \leq 0, \\ 0 & \text{elsewhere} \end{cases} \]

Hence the exact solution is \( u(t, x) = u_0(x - t) \), i.e. a “hump” of height and width one, traveling to the right with speed one.

The initial data for the FDS are given by

\[ v_{m}^{0} = \begin{cases} 1 & \text{if } -1 \leq m \cdot h \leq 0, \\ 0 & \text{elsewhere} \end{cases} \]
Example: Consistency $\nRightarrow$ Convergence

Figure: Illustration of how the exact solution propagates; it is one in the band, and zero outside the band. The initial condition for the FDS are zeros everywhere, except in the four points $v^0_0$, $v^0_{-1}$, $v^0_{-2}$, and $v^0_{-3}$, where it is one.

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Stability — The “Missing” Property

If a scheme is convergent, then as $v^n_m$ converges to $u(t,x)$, then $v^n_m$ is bounded in some sense; this is the essence of stability.

For almost all schemes there are restrictions on the way $h$ and $k$ can be chosen so that the particular scheme is stable. A stability region is any bounded non-empty region of the first quadrant of $\mathbb{R}^2$ that has the origin as an accumulation point:

Figure: Illustration of how the FDS solution propagates. In particular we have that $v^n_m \equiv 0$, $\forall m > 0$, $n \geq 0$. Hence, $v^n_m \not\rightarrow u(t_n,x_m)$ for $(t_n,x_m)$ in the part of the band strictly in the right half plane — $u$ is one there, but $v^n_m$ is zero, no matter how much we refine the grid.

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Definition (Stable Scheme)

A finite difference scheme $P_{k,h}v^n_m = 0$ for a first-order equation is stable in a stability region $\Lambda$ if there is an integer $J$ such that for any positive time $T$, there is a constant $C_T$ such that

$$ h \sum_{m=-\infty}^{\infty} |v^n_m|^2 \leq C_T h \sum_{j=0}^{J} \sum_{m=-\infty}^{\infty} |v^j_m|^2 $$

for $0 \leq nk \leq T$, with $(k,h) \in \Lambda$. 

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Stability — Notation

The quantity

$$\|w\|_h = \left[ h \sum_{m=-\infty}^{\infty} |w_m|^2 \right]^{1/2}$$

is the $L^2$ norm of the grid function $w$, and is a measure of the size (energy) of the solution. — The multiplication by $h$ is needed so that the norm is not sensitive to grid refinements (the number of points increase as $h \to 0$).

With this notation, the inequality in the definition can be written

$$\|v^n\|_h \leq \left[ C_T \sum_{j=0}^{J} \|v^j\|_h^2 \right]^{1/2} \quad \iff \quad \|v^n\|_h \leq C_T \sum_{j=0}^{J} \|v^j\|_h$$

The inequality expresses a limit (in terms of energy) of how much the solution can grow. Typically $J = (n-1)$ for an $n$-step scheme.

Checking for Stability... Using the Definition

Checking whether $\|v^n\|_h \leq C_T \sum_{j=0}^{J} \|v^j\|_h$ holds for a particular scheme directly from the definition can be a formidable task.

Example-1.5.1 in Strikwerda performs this test for the forward-time-forward-space scheme; the analysis takes up a good page of algebraic manipulations... We will return to this issue very soon with better tools in hand.

We note that there is a strong relation between the Stability of Finite Difference Schemes, and the Well-Posedness of PDEs (IVPs).

Well-Posedness for the IVP — Definition

Definition (Well-Posed IVP)

The initial value problem for the first-order partial differential equation $Pu = 0$ is well-posed if for any time $T \geq 0$, there exists a constant $C_T$ such that any solution $u(t, x)$ satisfies

$$\int_{-\infty}^{\infty} |u(t, x)|^2 \, dx \leq C_T \int_{-\infty}^{\infty} |u(0, x)|^2 \, dx$$

for $0 \leq t \leq T$.

All these concepts, consistency, well-posedness, stability, and convergence come together in the Lax-Richtmyer equivalence theorem.
Condition for Stability

We now turn our attention to the key stability criterion for hyperbolic PDEs.

In last lecture we saw some numerical evidence of the leapfrog scheme (applied to \( u_t + au_x = 0, \ a = 1 \)) breaking down when \( \lambda > 1 \).

The condition \( |a\lambda| < 1 \) is necessary for stability of many explicit FDS.

An explicit scheme is a scheme that can be written as

\[
v_{m+1}^{n+1} = \sum_{n' \leq n} v_{m'}^{n'}
\]

Implicit schemes, where the sum may contain terms with \( n' = n + 1 \), will be discussed soon.

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The Courant-Friedrichs-Lewy Condition

The following result covers all one-step schemes we have seen so far:

Theorem (The CFL Condition)

For an explicit scheme for the hyperbolic equation

\[
u_t + au_x = 0
\]

of the form

\[
v_{m+1}^{n+1} = \alpha v_{m+1}^{n} + \beta v_{m}^{n} + \gamma v_{m-1}^{n}
\]

with \( \lambda = k/h \) held constant, a necessary condition for stability is the Courant-Friedrichs-Lewy (CFL) condition,

\[|a\lambda| \leq 1.\]

For systems of equations for which \( \vec{v} \) is a vector and \( \alpha, \beta, \) and \( \gamma \) are matrices, we must have \( |a_i\lambda| \leq 1 \) for all eigenvalues \( a_i \) of the matrix \( A \).

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Another Theorem

Courant, Friedrichs and Lewy also proved the following theorem:

Theorem

There are no explicit, unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations.

One way of thinking about these theorems is to define the **numerical speed of propagation** as \( \lambda^{-1} = h/k \), and note that a necessary condition for the stability of a scheme is

\[
\lambda^{-1} \geq |a|.
\]

This guarantees that the FDS can propagate information (energy) at least as fast as the PDE.

\( \lambda^{-1} \) is the "speed limit" on the grid; which explains why (with \( a = 1 \)), we saw the breakdown of the leapfrog scheme when \( \lambda > 1 \iff \lambda^{-1} < 1 \).

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Convergence and Consistency
Stability
Definitions
The Lax-Richtmyer Equivalence Theorem
The Courant-Friedrichs-Lewy Stability Condition

Homework #1 — Due 2/9/2018, 12:00pm, GMCS-587

- **Strikwerda-1.3.1** — Numerical
- **Strikwerda-1.4.1** — Theoretical; but use software for Taylor expansions*.
- **Strikwerda-1.5.1** — Theoretical; but use software for Taylor expansions.

* In matlab, try:

```matlab
>> syms f(t,x) k h
>> taylor( f(t+k,x+h), [h,k], 'ExpansionPoint', [0,0], 'Order', 5)
... and ponder what the output means. In the long run you will save a lot of time if you can parse that output.
```

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